# Velocities Induced by Semi-Infinite Helical Vortex Filaments

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## Abstract

THE purpose of the synoptic is to present an efficient method of calculating the axial and radial components of the velocities induced by semi-infinite helical vortex filaments at any point in space. Since closed analytical solutions for these velocities do not exist, it is necessary to perform numerical integrations in order to calculate them. It will be shown that, by a combination of direct numerical integration of the exact integrand and exact analytical integration of its envelope functions, it is possible to obtain a very good approximation of the induced velocities. The use of this approximate method reduces significantly the required computing time. This reduction is important in the case of aerodynamic analyses where a very large number of calculations of the induced velocities is required. Such cases include, for example, free-wake analyses of rotors. In these analyses, the wake is usually divided into near and far wake regions. A natural description of the far wake is by semi-infinite helical vortex filaments.[1,2] Thus, it is necessary to calculate the velocities induced by the semi-infinite helical vortex filaments at many points in space (typical examples include several thousand calculations of the velocities induced by semi-infinite helical vortex filaments at different points in space) and the advantage of having an efficient method is clear.

## Contents

The semi-infinite helical vortex filament starts at the point $j$, which is defined by its polar coordinates $r_j$, $\phi_j$, and $z_j$. The pitch of the helix is constant and equal to $p_j$. It is assumed that the circulation along the vortex filament is also constant and equal to $\Gamma_j$.

The purpose is to calculate the velocity induced at the point $q(r_q,\phi_q,z_q)$ by the above described semi-infinite helical vortex filament. The velocity $\bar{v}_q$ is defined by its components,

$$\bar{v}_q = v_r^q\hat{e}_r + v_\phi^q\hat{e}_\phi + v_z^q\hat{e}_z \tag{1}$$

In the case of rotating blades, the circumferential component of the induced velocity should be added to a relatively large component of a circumferential velocity due to the blade rotation about the shaft. Therefore, in most of the practical cases, the contribution of the circumferential component of the induced velocity can be neglected. This is the reason that the rest of the synoptic will deal only with the axial and radial components of the induced velocity.

For convenience, influence coefficients will be defined as follows:

$$v_r^q = (\Gamma_j/r_j)I_{r,j}^q \tag{2a}$$

$$v_z^q = (\Gamma_j/r_j)I_{z,j}^q \tag{2b}$$

where $I_{r,j}^q$ and $I_{z,j}^q$ are nondimensional influence coefficients that determine the velocities induced by the $j_{\text{th}}$ helical vortex filament at the point $q$, in the radial and axial directions, respectively.

By applying Biot-Savart law, the following expressions for the influence coefficients are obtained:

$$I_{r,j}^q = \frac{1}{4\pi} \times \int_0^\infty \left[\frac{(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta)\cos(\phi_j+\eta-\phi_q)+\tilde{p}\sin(\phi_j+\eta-\phi_q)}{d^3}\right]\mathrm{d}\eta \tag{3a}$$

$$I_{z,j}^q = \frac{1}{4\pi}\int_0^\infty \left[\frac{1-\tilde{r}_q\cos(\phi_j+\eta-\phi_q)}{d^3}\right]\mathrm{d}\eta \tag{3b}$$

where

$$\tilde{z}_q = z_q/r_j; \quad \tilde{r}_q = r_q/r_j$$
$$\tilde{p} = p_j/2\pi r_j; \quad \tilde{z}_j = z_j/r_j \tag{4}$$

and:

$$d = \left[\tilde{r}_q^2 + 1 - 2\tilde{r}_q\cos(\phi_j+\eta-\phi_q) + (\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta)^2\right]^{1/2} \tag{5}$$

In relation to the calculation of $I_{r,j}^q$, the following functions are defined:

$$\tilde{I}_r', \tilde{I}_r'' = \pm\frac{1}{4\pi}\frac{(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta)}{[\tilde{r}_q^2+1\mp 2\tilde{r}_q+(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta)^2]^{3/2}} \tag{6}$$

It can be shown that $\tilde{I}_r'$ and $\tilde{I}_r''$ form lower and upper bounds of the integrand of $I_{r,j}^q$ defined by Eq. (3a).

Similarly, in relation to the calculation of $I_{z,j}^q$, functions $\tilde{I}_z'$ and $\tilde{I}_z''$ are defined as

$$\tilde{I}_z', \tilde{I}_z'' = \frac{1}{4\pi}\frac{(1\mp\tilde{r}_q)}{[\tilde{r}_q^2+1\mp 2\tilde{r}_q+(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta)^2]^{3/2}} \tag{7}$$

As in the case of the radial component, the integrand of $I_{z,j}^q$ defined by Eq. (3b) is bounded by $\tilde{I}_z'$ and $\tilde{I}_z''$.

The idea now is to use the envelope functions in order to obtain approximations of the influence coefficients. These approximations are expressed by the following equations:

$$I_{r,j}^q \cong \int_0^{\eta_r}\tilde{I}_r\mathrm{d}\eta + k_r\int_{\eta_r}^\infty(\tilde{I}_r'+\tilde{I}_r'')\mathrm{d}\eta \tag{8a}$$

$$I_{z,j}^q \cong \int_0^{\eta_r}\tilde{I}_z\mathrm{d}\eta + k_z\int_{\eta_z}^\infty(\tilde{I}_z'+\tilde{I}_z'')\mathrm{d}\eta \tag{8b}$$

where $\tilde{I}_r$ and $\tilde{I}_z$ are the integrands defined in Eqs. (3a) and (3b).

The advantage of using the approximations is that the integrations of the envelope functions have closed analytical solutions,

$$\int_{\eta_r}^\infty \tilde{I}_r'\mathrm{d}\eta = -\frac{1}{4\pi}\frac{1}{\tilde{p}[(1-\tilde{r}_q)^2+(\tilde{z}_q-z_j-\tilde{p}\eta_r)^2]^{1/2}} \tag{9a}$$

$$\int_{\eta_r}^\infty \tilde{I}_r''\mathrm{d}\eta = \frac{1}{4\pi}\frac{1}{\tilde{p}[(1+\tilde{r}_q)^2+(\tilde{z}_q-z_j-\tilde{p}\eta_r)^2]^{1/2}} \tag{9b}$$

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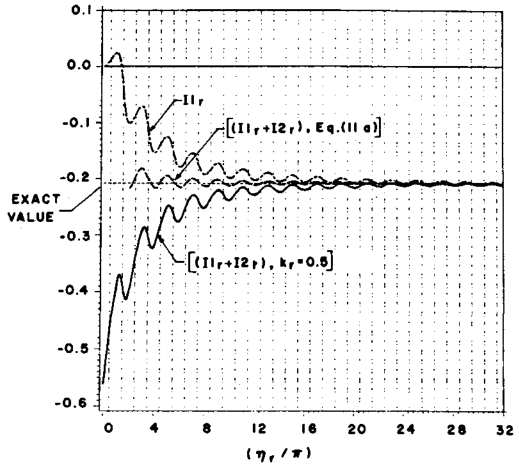
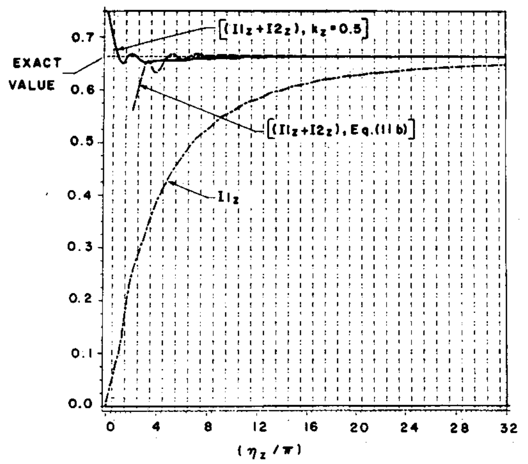



a) Radial component.




b) Axial Component.

**Fig. 1 Approximations of the influence coefficients as a function of the integration distance $[\tilde{r}_q = 0.5;\ (\phi_j - \phi_q) = \pi/2;\ (\tilde{z}_q - \tilde{z}_j) = -0.45;\ \tilde{p} = 0.06]$.**

$$\int_{\eta_z}^{\infty} \tilde{I}_z' \mathrm{d}\eta = \frac{1}{4\pi}\frac{1}{(1-\tilde{r}_q)\tilde{p}}\left[1+\frac{(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta_z)}{[(1-\tilde{r}_q)^2+(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta_z)^2]^{1/2}}\right] \tag{10a}$$

$$\int_{\eta_z}^{\infty} \tilde{I}_z'' \,' \mathrm{d}\eta = \frac{1}{4\pi}\frac{1}{(1+\tilde{r}_q)\tilde{p}}\left[1+\frac{(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta_z)}{[(1+\tilde{r}_q)^2+(\tilde{z}_q-\tilde{z}_j-\tilde{p}\eta_z)^2]^{1/2}}\right] \tag{10b}$$

In Fig. 1a, $I1_r$ and the sum $(I1_r + I2_r)$ for a certain case are described as functions of $\eta_r$. $I1_r$ and $I2_r$ are the first and second terms, respectively, on the right side of Eq. (8a). These functions are compared with the exact value of the influence coefficient $\tilde{I}^q_{r,j}$ (this exact value is obtained by direct integration in the interval $0 \leq \eta \leq 200\pi$). At first, a value of $k_r = 0.5$ was applied. This means that the approximation includes an average between the two envelope functions. This value gave very good results for approximations of the axially induced velocity in Ref. 3. It is shown that $I1_r$ and $(I1_r + I2_r)$ approach the exact value of the influence coefficient from above and below, respectively.

In Fig. 1b, $I1_z$ and the sum $(I1_z + I2_z)$ are described as functions of $\eta_z$. Similar to the case of the radial component, $I1_z$ and $I2_z$ are the first and second terms on the right side of Eq. (8b). It is shown that, while the asymptotic convergence of $I1_z$ to the exact value of the influence coefficient $I^q_{z,j}$ is slow, the convergence of $(I1_z + I2_z)$ (for $k_z = 0.5$) to the exact value is very fast.

Figure 1 shows that, while the approximation for the axial component of the induced velocity is very good, the approximation for the radial component does not yield clear improvement. Such improvement can be obtained by using a more sophisticated method to determine the factors $k_r$ and $k_z$. Since the period of the oscillations of the integrands is $2\pi$, $k_r$ and $k_z$ will be determined based on the behavior of the integration at the last $2\pi$ rad. Thus:

$$k_r = \left(\int_{\eta_r-2\pi}^{\eta_r} \tilde{I}_r \mathrm{d}\eta\right) \Big/ \left(\int_{\eta_r-2\pi}^{\eta_r} (\tilde{I}_r' + \tilde{I}_r'')\mathrm{d}\eta\right) \tag{11a}$$

$$k_z = \left(\int_{\eta_z-2\pi}^{\eta_z} \tilde{I}_z \mathrm{d}\eta\right) \Big/ \left(\int_{\eta_z-2\pi}^{\eta_z} (\tilde{I}_z' + \tilde{I}_z'')\mathrm{d}\eta\right) \tag{11b}$$

The approximations of the influence coefficients $[(I1_r + I2_r)$ and $(I1_z + I2_z)]$ when Eqs. (11) are used, are also presented in Fig. 1. It can be seen that in all cases the convergence to the exact value is faster with Eqs. (11). In the case of the problematic radial influence coefficient (Fig. 1a), there is a significant improvement of the convergence. In the case of the axial component (Fig. 1b), there is no clear improvement in the convergence due to the use of Eq. (11b), compared to a constant value of $k_z = 0.5$, but still convergence in both cases is relatively fast compared to simple direct integration.

It is apparent from the previous results that the convergence of the influence coefficients is usually oscillatory with a period of $2\pi$. Inspection of the equations indicates that these oscillations are mainly a result of the oscillatory behavior of the distance from the point where the induced velocity is calculated to a point on the semi-infinite helical vortex filament. Local extremum points of this distance occur at the neighborhood of [see also Eq. (5)],

$$\eta = \phi_q - \phi_j + (i-1)\pi \quad i \geq 1 \tag{12}$$

In Fig. 1 ordinate lines at these discrete values of $\eta$ ease the determination of the values of the different approximations at these points.

It is clear from Fig 1a that taking $\eta_r$ equal to the discrete values given by Eq. (12) gives very good approximations of the radial influence coefficients when Eqs. (8a) and (11a) are used. Concerning the axial influence coefficients, taking $\eta_z$ equal to the discrete values indicated shows only slight improvements. Similar results for other cases are shown in Ref. 4.

Investigation of the results (for more details, see Ref. 4) indicate that the values of the coefficients $k_r$ and $k_z$ at the discrete points [given by Eq. (12)] are practically constant and equal to 0.25 and 0.5, respectively. Therefore, it seems that, instead of using Eqs. (11), one may choose a value of $k_r = 0.25$ and $k_2 = 0.5$ and calculate the approximations at the discrete points given by Eq. (12).

## References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